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Optimal trading from minimizing the period of bankruptcy risk

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Abstract. Assuming that financial markets behave similar to random walk processes we derive a trading strategy with variable investment which is based on the equivalence of the period of bankruptcy risk and the risk to profit ratio. We define a state dependent predictability measure which can be attributed to the deterministic and stochastic components of the price dynamics. The influence of predictability variations and especially of short term inefficiency structures on the optimal amount of investment is analyzed in the given context and a method for adaptation of a trading system to the proposed objective function is presented. Finally we show the performance of our trading strategy on the *DAX* and *S&P 500* as examples for real world data using different types of prediction models in comparison.

PACS. 89.90.+n Other topics in areas of applied and interdisciplinary physics – 02.50.Ey Stochastic processes – 05.45.Tp Time series analysis

1 Introduction

The theory of efficient markets denies the possibility for finding a profitable trading system which is able to forecast future market behavior [3]. In contrast to that theory there are successes of expert traders and also of trading systems which can only be explained by temporary inefficiency margins in the market and consequently by information not incorporated in the current market prices [17]. If such periods of inefficiency and the associated fluctuations of predictability are not recognized by the trading models they cause undesired variations in their performance. Further, almost all trading and portfolio selection strategies take into account that beside maximization of profit a very important objective for optimizing trading systems with finite reserve fund is also a simultaneous minimization of risk [11,14].

Inspired by these developments we present a method for training an appropriate trading system for direct estimation of the optimal amount of investment, motivated by the objective to minimize the period of bankruptcy risk. We show that this objective is related to a common performance measure in finance and fulfills the requirements stated above. Despite new insights in the stochastic nature of price fluctuations, in particular that they obey truncated Lévy flights [7–10] and not Wiener processes, the latter is still very common in portfolio analysis techniques [2]. The reason is that all Lévy stable processes with $\alpha < 2$ have infinite variance which leads to fundamental problems when applied to finance, where the second moment is related to risk estimation [6]. Therefore, we also make use of the Gaussian assumption in order to embed our work in that field.

Our derivation of the trading strategy is based on the concept of a random walk with state dependent drift and variance for the description of profit evolution. The assumption of state dependency takes into account significant deviations from stationarity of market dynamics [5]. Here, this leads to time dependent strengths of stochastic and deterministic components of the random walk process. Based on the probability for gaining or loosing money a predictability measure will be defined. A reasonable objective function for finding the optimal amount of investment depending on the current market predictability such that profit and risk are optimized could be the risk to profit ratio. This measure is quite similar to the inverse Sharpe Ratio [14], neglecting the return of a risk–less security. We show that this objective is equivalent to the minimization of the period of bankruptcy risk and finally we present a method for training a trading system which directly estimates the optimal investment and the expected return. In the context of the results on the *DAX* and *S&P 500* with a discrete state space (DSS) model and a radial basis function (RBF) neural network as predictive models, we emphasize the necessity of using trading models with variable investments in order to react to changing market predictability and to reduce the variance in gained profit.

2 Theory

Previous work demonstrated a close relation between diffusion processes in physical systems and the evolution of prices in financial markets [1,4,16]. On that basis we assume that also the evolution of profit $g(\mathbf{x}_n)$ can be

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described by a discrete random walk with state dependent drift and variance terms:

$$g(\mathbf{x}_n) = \bar{\mu}(\mathbf{x}_n)\tau + \sigma(\mathbf{x}_n)\Delta X. \tag{1}$$

τ denotes a constant time interval between successive trading events, $\bar{\mu}(\mathbf{x}_n)$ the local profits and $\sigma(\mathbf{x}_n)^2$ the local variances. The series $\{\mathbf{x}_n\}_{n=1}^N$ of market state vectors incorporates all available informations which are used for generating trading decisions. The market state vector can *e.g.* be constructed by embedding [15] the time series of returns r_n into a space of embedding dimension n_e: $\mathbf{x}_n = \{r_{n-1}; \ldots ; r_{n-n_\mathrm{e}}\} \in \mathrm{R}^{n_\mathrm{e}}$. The returns are determined by the logarithmic differences of prices p_n, so $r_n = \log(p_{n+1}) - \log(p_n)$.

As stated in the introduction we assume a Wiener process for the random variable $\Delta X = \Psi\sqrt{\tau}$ with

$$\mathrm{Prob}(\Psi \in [\phi, \phi + \mathrm{d}\phi]) = \frac{\mathrm{e}^{-\phi^2/2}}{\sqrt{2\pi}}\mathrm{d}\phi =: \varphi(\phi)\mathrm{d}\phi. \tag{2}$$

In this framework the stochastic and deterministic components of the process can be identified with the terms in equation (1). The stochastic component incorporates all non-predictable fluctuations caused by new information entering the market or by irrationality in the investors' decision processes. The deterministic component is due to inefficiencies in the financial market which are assumed to be detected by an appropriate trading system. The latter would cause a positive drift in the cumulated profit. The relative strengths of both components will in general exhibit a very complex time varying behavior.

A local predictability ω_n, based on the probability (or hit quota) $\eta(\mathbf{x}_n) = \mathrm{Prob}(g(\mathbf{x}_n) > 0)$ of obtaining a positive profit at time $n\tau$, can be defined as follows:

$$\omega(\mathbf{x}_n) = 2\eta(\mathbf{x}_n) - 1 = 2\Phi_0\left(\frac{\bar{\mu}(\mathbf{x}_n)}{\sigma(\mathbf{x}_n)}\sqrt{\tau}\right), \tag{3}$$

with $\Phi_0(z) = \int_0^z \varphi(\phi)\mathrm{d}\phi = \mathrm{erf}(z) - \frac{1}{2}$. It is obvious that the interplay of variations in the both components mentioned above affect the behavior of predictability. Other definitions, inspired *e.g.* by information theory can also be found in literature [12]. Definition (3) limits the predictability measure to $\omega(\mathbf{x}_n) \in [-1; 1]$ whereby negative values can only be due to systematically wrong predictions of the trading system, which, by inversion, would also exhibit predictive power.

Up to now only the local behavior of the stochastic process at state $\mathbf{x}_n$ is characterized. With regard to the purpose of trading optimization on the global dynamics the mean process with respect to the state space (and not to the realizations of the stochastic process) can be expressed as

$$\bar{g} = \int g(\mathbf{x})p(\mathbf{x})\mathrm{d}\mathbf{x} = \bar{\mu}\tau + \sigma\Delta X, \tag{4}$$

with mean profit $\bar{\mu} = \int \bar{\mu}(\mathbf{x})p(\mathbf{x})\mathrm{d}\mathbf{x}$, mean variance $\sigma^2 = \int \sigma(\mathbf{x})^2 p(\mathbf{x})\mathrm{d}\mathbf{x}$ and a state density function $p(\mathbf{x})$. With probability $\int_{-\lambda}^{\lambda} \varphi(\phi)\mathrm{d}\phi$ the cumulated profit G_n after n successive trading events will be found in the confidence region $G_n \in [C_n(-\lambda); C_n(\lambda)]$, with $C_n(\lambda) = \bar{\mu}n\tau + \lambda\sigma\sqrt{n\tau}$. This interval is symmetric around the evolution of cumulated mean profit and parameter λ determine the width of the interval in units of σ.

At the beginning of trading the confidence region has a significant component of negative profit (loss of money) which signifies a certain time of risk t_0 where G_n could be negative. This corresponds to a substantial probability to go bankrupt if no sufficient amount of reserve fund G_f is available. The reserve fund G_f is necessary with highest probability after time t_f. Time of risk and reserve fund are given by

$$t_0 = \lambda^2\frac{\sigma^2}{\bar{\mu}^2} \tag{5}$$

$$t_\mathrm{f} = \frac{t_0}{4} \text{ and } G_\mathrm{f} = -C_{t_\mathrm{f}}(-\lambda) = \bar{\mu}\frac{t_0}{4}\cdot \tag{6}$$

Minimization of t_0 means minimization of risk to profit ratio over the specified trading period. This fulfills exactly the requirements for an optimal investment signal. Therefore, using the confidence level $\lambda = 1$, we define t_0 as the objective function for the optimization process of the trading system. Neglecting the return of a risk-less security this objective corresponds to the inverse squared Sharpe ratio [14]. Our derivation shows that the usage of the Sharpe ratio as a performance measure can be motivated and interpreted nicely within the framework of stochastic processes.

For the final form of objective function (5) we have to specify how the trading decision influences the profit. We assume a trading system is given which provides an investment signal $a(\mathbf{x}_n)$ that quantifies the volume of stock order depending on a market state vector $\mathbf{x}_n$. The resulting profit $g(\mathbf{x}_n)$ with respect to the return r_n is then given by $g(\mathbf{x}_n) = a(\mathbf{x}_n)r_n$. This yields to:

$$\hat{t}_0 = \frac{\frac{1}{T}\sum_{n=1}^N a(\mathbf{x}_n)^2\left(r(\mathbf{x}_n) - \bar{r}(\mathbf{x}_n)\right)^2}{\left(\frac{1}{T}\sum_{n=1}^N a(\mathbf{x}_n)r(\mathbf{x}_n)\right)^2}\cdot \tag{7}$$

Mean local return $\bar{r}(\mathbf{x}_n)$ as well as sign and magnitude of the investment signal $a(\mathbf{x}_n)$ remain to be estimated by two successive steps with appropriate models and methods. For approximation of the mean local return, any type of models can be optimized using standard algorithms like least–squares techniques with gradient descent. Assuming a given return model $\tilde{r}(\mathbf{x}_n) = h(\boldsymbol{\Xi}, \mathbf{x}_n)$, with parameter vector $\boldsymbol{\Xi}$, adaptation of the trading system $a(\mathbf{x}_n) = f(\boldsymbol{\Theta}, \mathbf{x}_n)$ can be performed in a second step by gradient descent or other suitable methods on the objective function equation (7).

3 Models and parameter estimation

3.1 Discrete state space model

The first prediction model for our simulations is a discrete state space (DSS) model. Discretization is done by taking only the signs of each input component of the original state vector $\mathbf{x}_n$ which leads to 2^{n_e} possible combinations of signs. So, the transformation to the discrete state s_n of the DSS model at trading event n can be described by the mapping function q: $\mathrm{R}^{n_e} \rightarrow [1;\ldots;2^{n_e}]$ with $s_n = q(\mathbf{x}_n)$. Due to this discretization of state space the estimations of the model are given by the averages over all returns with respect to their state

$$\tilde{r}(\mathbf{x}_n) = \langle r(\mathbf{x}_{n'})\rangle_{q(\mathbf{x}_n)}, \tag{8}$$

and by an appropriate expression for the investments derived by the condition $\boldsymbol{\nabla}_{\boldsymbol{\Theta}}\hat{t}_0 = 0$

$$a(\mathbf{x}_n) = c\left\langle \frac{r(\mathbf{x}_{n'})}{(r(\mathbf{x}_{n'}) - \tilde{r}(\mathbf{x}_{n'}))^2}\right\rangle_{q(\mathbf{x}_n)}. \tag{9}$$

c is a constant and $\langle\cdot\rangle_{q(\mathbf{x}_n)}$ the average over all time steps n' whose state $s_{n'}$ is equal to state $s_n = q(\mathbf{x}_n)$.

3.2 RBF model

The radial basis function (RBF) neural network of Moody Darken type [13] has the functional form

$$\tilde{y}_n = \frac{\sum_i w_i g_i(\mathbf{x}_n)}{\sum_j g_j(\mathbf{x}_n)} \quad \text{with} \quad g_i(\mathbf{x}_n) = \mathrm{e}^{-(\mathbf{x}_n - \mathbf{z}_i)^2/2\sigma_i^2} \tag{10}$$

with parameters w_i, z_i and σ_i. In our application the network output $\tilde{y}_n$ represents the estimated mean return $\tilde{r}(\mathbf{x}_n)$ or the investment signal $a(\mathbf{x}_n)$. Training is performed as usually by an unsupervised adaptation of centers z_i and width σ_i of Gaussians g_i using K-means clustering, in case of the return model a linear matrix inversion technique finally adjusts the second layer weights w_i. For adaptation of the trading model $f(\boldsymbol{\Theta}, \mathbf{x}_n)$ a supervised gradient training using equation (7) is applied to the weights. Over-fitting is avoided by using only a small number of Gaussians and controlling the generalization performance on test data. The number of radial basis function centers $\mathbf{z}_i$ is set to be 2^{n_e} for reasons of comparability between the DSS model and the RBF network. That choice implies the same number of effective parameters in both types of models.

3.3 Parameter estimation

For approximation of the local mean return and the investment signal a gradient descent on the objective function (7) is used. The consideration of the constraint

$$\frac{1}{T}\sum_{n=1}^{N} |a(\mathbf{x}_n)| = A \stackrel{!}{=} \text{const.} \tag{11}$$

ensures a constant mean absolute investment despite local changes during the optimization procedure. The gradient of $\hat{t}_0$ with respect to the parameter vector $\boldsymbol{\Theta}$ is

$$\boldsymbol{\nabla}_{\boldsymbol{\Theta}}\hat{t}_0 = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{\nabla}_{\boldsymbol{\Theta}} a(\mathbf{x}_n)\left(\frac{2a(\mathbf{x}_n)\Delta r(\mathbf{x}_n)^2}{\hat{\mu}^2} - \frac{2\hat{t}_0 r(\mathbf{x}_n)}{\hat{\mu}} - \beta \mathrm{sgn}(a(\mathbf{x}_n))\right), \tag{12}$$

with $\Delta r(\mathbf{x}_n) = r(\mathbf{x}_n) - \tilde{r}(\mathbf{x}_n)$ and Lagrange multiplier

$$\beta = \frac{NA - \hat{t}_0\hat{\mu}\sum_{n=1}^{N} r(\mathbf{x}_n)\mathrm{sgn}(a(\mathbf{x}_n))\Delta r(\mathbf{x}_n)^{-2}}{\frac{1}{2}\hat{\mu}^2\sum_{n=1}^{N}\Delta r(\mathbf{x}_n)^{-2}}. \tag{13}$$

Table 1. The quantitative results of risk to profit ratios $\hat{t}_0$ are shown for simulations on time series of *DAX* and *S&P 500* close values using the DSS model, a RBF network and the respective constant investment versions (sgn).

		DAX		*S&P 500*	
		Train	Test	Train	Test
DSS	$\hat{t}_0 =$	35.1	2009.4	96.5	129.7
DSS (sgn)	$\hat{t}_0 =$	53.3	1233.1	126.9	154.5
RBF	$\hat{t}_0 =$	77.0	243.8	80.1	127.6
RBF (sgn)	$\hat{t}_0 =$	162.1	1411.4	103.3	451.5

4 Application to DAX and S&P 500

We demonstrate the performance of our trading strategy with two real financial time series and compare it to a constant investment strategy. Latter strategy gives a constant trading signal in the direction of expected return. Therefore, it does not use any volatility information. For reason of comparability the investment signals of both strategies are normalized to the same amount of mean absolute investment over the whole trading period. In the case of variable investment, the maximal amount of absolute investment is limited to five times its mean because also in real trading the possible investment is in general restricted to some maximum value.

For testing the generalization ability we use the first 70% of the data in the training procedure, the remaining patterns are excluded from training and used for the test purpose. The time series of returns are transformed to zero mean, so that all profit gained by the trading system is excess profit exceeding the possible profit by the simple usage of a buy-and-hold strategy.

The available time series of German Stock Market Index (*DAX*) consists of 2340 daily close prices of an almost 10 year period from 11/26/1990 until 03/17/2000. The time series of the Standard & Poor's 500 Index (*S&P 500*) consists of 5109 daily close prices of about a 20 year period from 02/01/1980 until 03/17/2000. In both cases we use an embedding dimension $n_e = 5$ for adaptation of the different types of prediction models. As stated above, this leads to the choice of 32 centers in the RBF network in order to ensure comparability between the models. Table 1

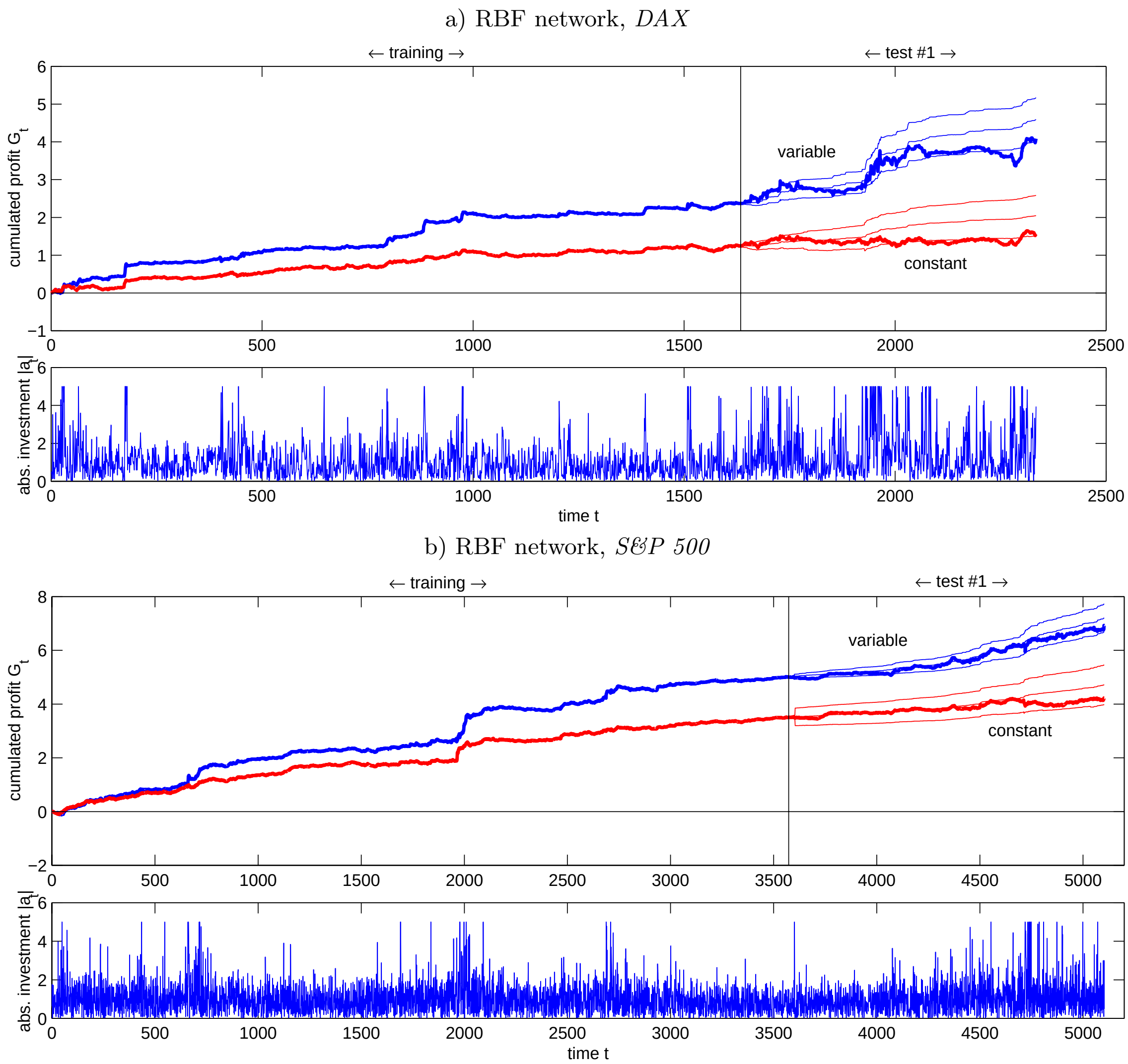

Fig. 1. Evolution of cumulated profit and absolute investments $|a(\mathbf{x}_n)|$ while trading with RBF networks as prediction models. (a) shows trading on the *DAX* time series, (b) on *S&P 500* time series, each in comparison to constant investment signals. Curves of expected profit evolution and the border of 95% confidence region ($\lambda = 2$) are added in order to demonstrate the correspondence between trading model and real market behavior.

shows the final risk to profit ratios $\hat{t}_0$ of both models and their constant investment versions.

Generally the performance of both types of models decreases if they are used on the test data sets. This concerns especially the simulation on the *DAX* which can be explained by changes in the fundamental market dynamics (non–stationarities) and by the crash and strong correction in the year 1998 which lies in the test data. Obviously the RBF model is able to handle these dynamical processes better than the DSS model. On the *S&P 500* both kinds of models shows quite similar $\hat{t}_0$ values with a small superiority of the RBF network. Figures 1a and b demonstrate the profit evolution of the RBF model on both data sets together with the expected profit evolution and the confidence curves for $\lambda = 2$. A very remarkable result is, that besides the more or less quiet market behavior also strong market movements, *e.g.* during the period of strong market corrections in 1998 at the *DAX* data, are estimated quite well. The comparison to the constant investment versions of our trading models shows a significant drop in performance in all cases, except the DSS model and *DAX* test data set. Latter can be explained by the difficulties of the DSS model in dealing with the market dynamic during the specified time region as specified above. Such situations can be detected by checking the confidence of the current profit evolution.

5 Summary

We derived an strategy with variable investments for trading in financial markets which are subject to changes in predictability of their dynamics. We motivated the proposed strategy by the equivalence of the period of bankruptcy risk and the risk to profit ratio which could be

derived in the framework of stochastic processes.
We demonstrated the performance of our trading strategy with a DSS model and a RBF neural network as prediction models on two real financial time series – the *DAX* and the *S&P 500* – and could show that the variable investment strategy and the used prediction models could mostly capture the main dynamical characteristics of the time series. Furthermore we provided a confidence estimation for an on-line evaluation of trading systems.

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